Merton's Partial Differential Equation and Fixed Point Theory

Franklin Lowenthal; Arnold Langsen; Clark T. Benson

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1. INTRODUCTION: A BRIEF VIEW OF CORPORATE FINANCE FOR MATHEMATICIANS. In the last twenty five years mathematical economics and finance has developed into an area rich in both its theoretical structure and the power and complexity of its applications. The basis for this subject is stochastic methodology, including stochastic calculus, stochastic differential equations, and optimal stochastic control. Topics such as martingale methods, optimal stopping, the modeling of uncertainty using a Wiener process, and Ito's Lemma are at the core of the theory. Applications to economics include stochastic capital theory, stochastic economic growth, the rational expectations hypothesis, and the competitive firm under price uncertainty. Applications to finance include the Black-Scholes option pricing model, portfolio rules, demand for index bonds, asset pricing, and the market risk adjustment in project valuation.

A fundamental principle of corporate finance is that the market value of the assets of a business is equal to the average expected cash flows generated in perpetuity, discounted to present value by a rate of return commensurate with the risk perceived by investors. Let A be the market value of the assets, let C denote the average expected cash flows, and let r_k be the rate of return required by investors for the risk class k. Then

$$A = \frac{C}{r_k}.$$

The cash flows are accumulated and are used to pay the promised contractual liabilities of the business; the balance accrues to the benefit of the shareholders. The contractual liabilities are in the form of debt, usually bonds or notes, with a promise to receive interest and principal on a specific maturity date. If the accumulated cash flows are not sufficient to pay on time and in full, the firm is in technical bankruptcy and the debtholders, by rights specified in their contract, can claim the residual assets of the firm. If we let B be the value of the claim of the debtholder and let D be the market value of debt, the claim can be expressed as

$$B = \min(D, A).$$

The debtholder says, "Either pay me what you promised or I'll claim the residual value of the liquidated assets." Therefore, we refer to B as a *contingent claim* on the assets of the firm.

The shareholders have no contractual arrangement and have a lower priority than the debtholder in the order of contingent claims to the assets of the firm. Thus, if we let E be the value of the claim of the shareholder, called *equity*, it can be expressed as

$$E = \max(A - D, 0).$$

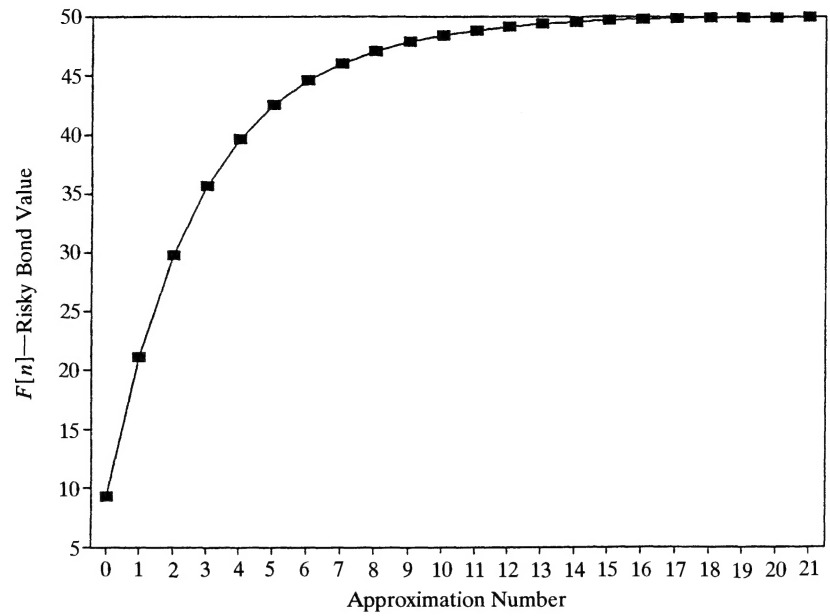

Figure 1. $F[n]$ Approximants. $F[n]$; $t = 36$, sigma = .5.

The claim of the shareholder differs from that of the debtholder in several ways. First, there is no contract. Second, the shareholder can claim the accumulated funds only after the debtholder claim has been satisfied. But third, the upside is unlimited by any maturity date or amount. Furthermore, the shareholder has limited liability: If the debt cannot be paid, the most that can be lost is the initial investment. We shall show that the purchase of a share of stock is the same as buying a *call option* on the assets of the firm.

Putting these concepts together we conclude that the market value of the assets of a firm is equal to the market value of its debt plus the market value of its outstanding shares of stock.

A call option is a contract that can be bought, sold, or assigned giving the owner the right, but not the obligation, to buy an asset at an agreed upon price at any time on or perhaps before a specified date in the future. The agreed upon exercise price is called the *strike price* and the maturity date is the *expiration date*. If, at maturity, the value of the underlying asset is not higher than the strike price, the option will not be exercised by a rational investor. That is, the value of a call option, CO, is a function of the strike price, SP, and the market value of the asset, A:

$$CO = \max(A - SP, 0).$$

A call option will be exercised if the value of the underlying asset is greater than the strike price; otherwise, a rational investor will walk away from the deal, losing only the price paid for the option. Thus, the terms of a call option are identical to the claims of a shareholder on the assets of the firm, where the strike price is the value of the debt obligation.

How does one put a price on the call option? The answer is more complex than one might imagine.

2. MERTON'S BOUNDARY VALUE PROBLEM. Building on the option pricing theorem of Black and Scholes [1], which suggests that the option pricing model can

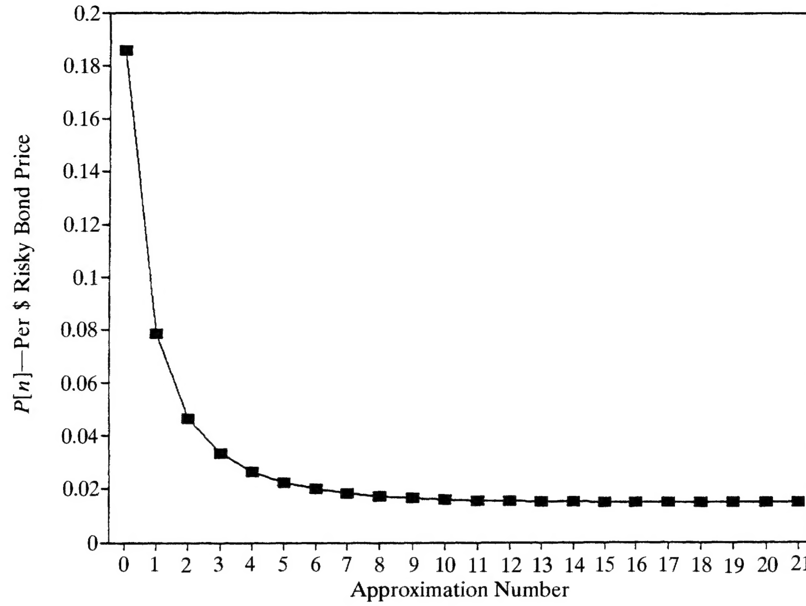

Figure 2. $P[n]$ Approximants. $P[n]$; $t = 36$, sigma $= .5$.

be used to price elements of corporate capital structure, Merton [8] developed a theory for pricing discount bonds when there is significant probability of default at maturity. He considered only the risk of default, and ignored risk due to unanticipated interest rate changes prior to maturity of the bond. In addition, Merton derived a risk structure of interest rates as a function of the debt-to-equity ratio (a measure of the riskiness of the assets of the firm) and the riskless debt rates.

Merton [8, p. 453] shows that the market value of a risky discount bond debt without coupon payments is the unique solution of the boundary value problem that consists of the parabolic partial differential equation

$$\tfrac{1}{2}\sigma^2 V^2 F_{VV} + \Omega V F_V - \Omega F - F_t = 0 \tag{1}$$

and the initial and boundary conditions:

$$F(0,t) = 0 \qquad t \geq 0 \tag{2a}$$

$$F(V,0) = \min(B,V) \tag{2b}$$

$$F(V,t) \leq V \qquad \text{or, equivalently,} \ \lim_{V\to\infty} F(V,t) = Be^{-\Omega t}. \tag{2c}$$

Here the independent variables V and t are, respectively, the market value of the firm and the time until maturity. The value B is the promised payment at maturity, while Ω is the riskless rate of return (for example, the rate paid by US Treasury bonds) and σ^2 is a measure of the volatility (variance) of the firm's assets. The boundary conditions are critical; Merton pointed out that they "distinguish one security from another (e.g., the debt of a firm from its equity)."

Merton solves his boundary value problem by deducing an isomorphic relationship between it and the boundary value problem for the market value of a certain type of call option on a non-dividend-paying common stock, the latter problem having already been solved in the literature. However, it is more interesting for our purposes to transform the boundary value problem for the market value of a risky

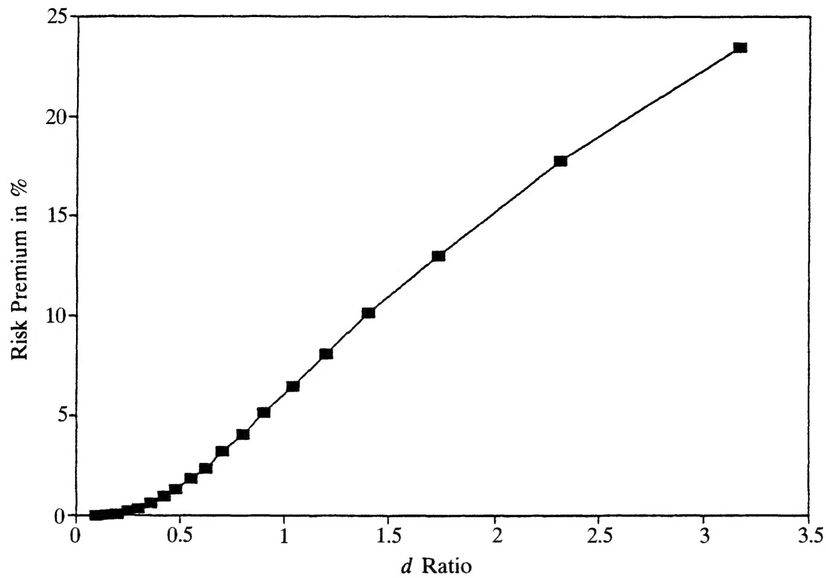

Figure 3. Risk premium v. d ratio. $t = 5$, sigma $= .3$.

discount bond debt into a classic problem in mathematical physics involving heat flow in an infinite insulated bar. The change of independent variable $V = e^W$ transforms (1) to

$$\frac{\sigma^2}{2} G_{WW} + \left(\Omega - \frac{\sigma^2}{2} \right) G_W - \Omega G - G_t = 0, \tag{3}$$

and the change of dependent variable $G(W, t) = e^{\alpha W + \lambda t} u(W, t)$, where $\alpha = \frac{1}{2} - \Omega / \sigma^2$ and $\lambda = -\Omega - \sigma^2 \alpha^2 / 2$, leads to the heat equation

$$\frac{\sigma^2}{2} u_{WW} = u_t. \tag{4}$$

The boundary conditions become

$$u(W, 0) = e^{-\alpha W} \min(B, e^W) = f(W) \tag{5a}$$

$$u(W, t) \to 0 \text{ as } t \to \infty. \tag{5b}$$

The solution of the heat equation (4) subject to the boundary conditions (5a) and (5b) may be obtained using the standard methods of separation of variables and Fourier integrals. It is

$$u(W, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f\left(W + s\sqrt{2\sigma^2 t}\right) e^{-s^2} \, ds. \tag{6}$$

Algebraic manipulations of (6) yield the solution to Merton's boundary value problem for a risky discount bond:

$$F(V, t) = B e^{-\Omega t} P(d), \tag{7}$$

where

$$d = \frac{B e^{-\Omega t}}{V} \tag{8}$$

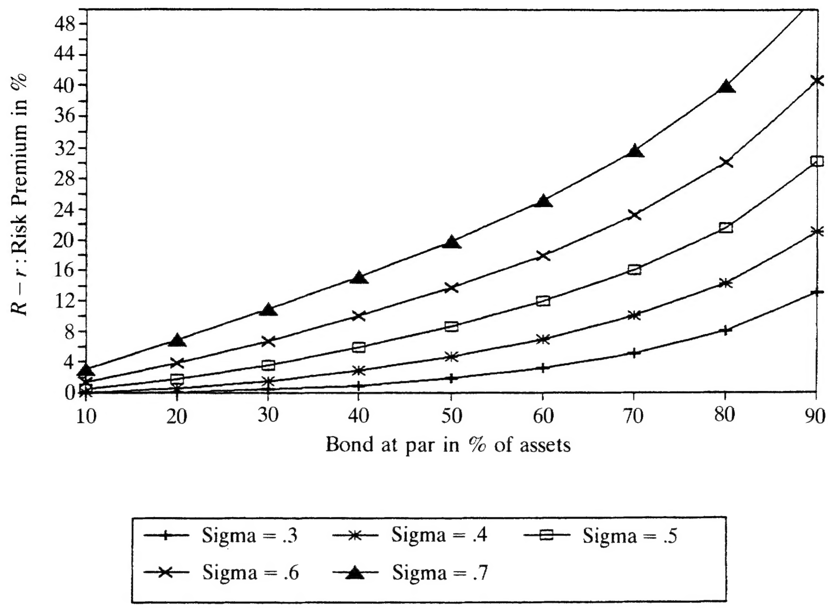

Figure 4. Risk premium v. par bond. $R - r$; $t = 5$.

is the *quasi debt-to-asset ratio* of the firm and

$$P(d) = N(h_2) + \frac{1}{d}N(h_1),$$

where $N(x)$ is the cumulative distribution function of the normal distribution,

$$h_1 = -\frac{\sigma}{2\sqrt{t}} + \frac{\ln d}{\sigma\sqrt{t}}, \tag{9}$$

and

$$h_2 = -\frac{\sigma}{2\sqrt{t}} - \frac{\ln d}{\sigma\sqrt{t}}. \tag{10}$$

The function $P(d)$ is the price of a dollar of risky debt. Merton computed the derivative of $P(d)$ and found

$$P'(d) = -N(h_1)/d^2. \tag{11}$$

Thus, $P(d)$ is a decreasing function on $[0, \infty)$. As $d \to \infty$, $P(d) \to 0$; L'Hospital's rule shows that $P(d) \to 1$ as $d \to 0^+$. Hence, the range of $P(d)$ is $(0, 1]$. This makes financial sense as the price of a dollar of risky debt should always be strictly less than a dollar!

Let b be the par value (face value) of the bond. If interest compounds continuously at the riskless rate of return Ω, then the proceeds at maturity of the bond are $b\exp(\Omega t)$. Substituting this value for B in (7) leads to $F(V, t) = bP(d) < b$ since $P(d)$ must be less than one. For example, if $\sigma^2 = .1$, $t = 5$, $\Omega = 10\%$, $b = 50$, and $V = 100$ (so $d = 50/100 = .5$, assuming that the proceeds at maturity are $B = 50\exp(.10 \times 5) = 82.436$), then $P(d) = .917$ and the market value at the time of issue is $F(100, 5) = 45.84$. The market debt-to-asset ratio is $45.84/100 = .4584$.

Merton concluded that the yield to maturity on the risky debt, provided that the firm does not default, is given by

$$R = \Omega - \frac{1}{t}\ln(P(d)). \tag{12}$$

In our example, $R = .10 + .01736 = .11736$; the amount by which R exceeds Ω is called the *risk premium*. Replacing Ω by R in $b\exp(\Omega t)$ leads to the value $50\exp(.11736 \times 5) = 89.911$. What is the market value of a risky bond paying this greater value at maturity?

Note that $d = B\exp(-\Omega t)/V = 89.911\exp(-.10 \times 5)/100 = 54.5337/100 = .545337$. Since d has increased from .5 to .545337, $P(d)$ has decreased and the resulting value of F is again less than 50 (but greater than 45.84). The resulting market debt-to-asset ratio is larger than .4584 but still less than .50. What if the company seeks to achieve a specified *market* debt-to-asset ratio of .50? Can we iteratively find a limiting value $\overline{B}$ that, when substituted into equation (7) with d given by equation (8), leads to a market value of the debt-to-asset ratio of $50/100 = .50$? The method of successive approximations may be used to find the fixed points of a contraction map and determine the value $\overline{B}$ in such a way that at each stage the successive market values satisfy Merton's boundary value problem.

3. THE METHOD OF SUCCESSIVE APPROXIMATIONS. It is convenient to begin our iterations by choosing $d_0 = 0$. Since $P(0) = 1$, substituting $d_0 = 0$ into (12) yields $R_1 = \Omega$. Then $B_1 = b\exp(R_1 t) = b\exp(\Omega t)$. Now we can compute the next iteration $d_1 = B_1\exp(-\Omega t)/V = b\exp(\Omega t)\exp(-\Omega t)/V = b/V$. Substituting this value for d in (12) yields the next iteration $R_2 = \Omega - (1/t)\ln(P(b/V))$; then $B_2 = b\exp(R_2 t)$ and $d_2 = B_2\exp(-\Omega t)/V$. In general, with $d_0 = 0$ we have the following iterations for $n \geq 1$:

$$R_n = \Omega - (1/t)\ln(P(d_{n-1})) \tag{13a}$$

$$B_n = b\exp(R_n t) \tag{13b}$$

$$d_n = B_n\exp(-\Omega t)/V. \tag{13c}$$

Because of (7), we define the iteration market prices of the risky bond debt in terms of the successive quasi debt-to-asset ratios d_n by

$$F_n(V,t) = B_n\exp(-\Omega t)P(d_n). \tag{14}$$

In the case of the simple example, it appears that all of $\{R_n\}$, $\{B_n\}$, $\{d_n\}$, and $\{F_n\}$ are monotonically increasing convergent sequences, with F_n approaching the par value b as its limit. In fact, the sixth iteration yields $R_6 = 12.222\%$, $B_6 = 92.1256$, $d_6 = .55877$, and $F_6 = 49.9999$; see Table 1. The convergence is less rapid for larger values of the product $\sigma\sqrt{t}$.

We now prove that the successive iterations converge. We assume only that the par value b of the bonds is less than the market value of the total assets V of the firm (otherwise, it would be in bankruptcy!) We begin by rewriting (13b), (13c), and (14) in a more convenient form. Since $\exp(R_n t) = \exp(\Omega t)/P(d_{n-1})$, we have

$$B_n = b\exp(\Omega t)/P(d_{n-1}), \tag{15a}$$

$$d_n = (b/V)(1/P(d_{n-1})), \tag{15b}$$

and

$$F_n = b(P(d_n)/P(d_{n-1})). \tag{15c}$$

TABLE 1 AN EXAMPLE OF SUCCESSIVE APPROXIMATIONS

assets = 100			time = 5		riskless = 0.1		sigma = 0.3162			
	eq(13b)		eq(13c)	eq(9)	eq(10)		eq(14)		eq(13a)	
iteration n	$B(n)$	$b(n)$	$d(n)$	$h1(n)$	$h2(n)$	$P(n)$	$F(n)$	$q(n+1)$	$R(n+1)$	$M(n+1)$
0	82.4361	50.0000	0.5000	−1.3338	0.6267	0.9168	45.8420	0.0174	11.7365%	0.4584
1	89.9133	54.5352	0.5454	−1.2110	0.5039	0.8999	49.0785	0.0211	12.1085%	0.4908
2	91.6016	55.5592	0.5556	−1.1847	0.4776	0.8960	49.7832	0.0220	12.1954%	0.4978
3	92.0006	55.8012	0.5580	−1.1786	0.4715	0.8951	49.9483	0.0222	12.2161%	0.4995
4	92.0958	55.8589	0.5586	−1.1771	0.4700	0.8949	49.9876	0.0222	12.2211%	0.4999
5	92.1185	44.8727	0.5587	−1.1768	0.4697	0.8948	49.9970	0.0222	12.2222%	0.5000
6	92.1240	55.8760	0.5588	−1.1767	0.4696	0.8948	49.9993	0.0222	12.2225%	0.5000

$q = -1/t * \ln(P(D))$; M is the ratio of the market value of a risky bond to the market value of the assets.

Lemma 1. *The market value of a dollar of risky bond debt, $P(d)$, satisfies the functional equation*

$$P(1/d) = dP(d). \tag{16}$$

Proof: Since $\ln(1/d) = -\ln(d)$, replacing $\ln d$ by $\ln(1/d)$ in (9) and (10) transforms h_2 into h_1 and h_1 into h_2. Hence $P(1/d) = N(h_2(1/d)) + dN(h_1(1/d)) = N(h_1) + dN(h_2) = dP(d)$. ■

Theorem 1. *The sequence of quasi debt-to-asset ratios $\{d_n\}$ is strictly increasing.*

Proof: Proceed by mathematical induction. We have $d_0 = 0$ and $d_1 = b/V$, so $d_0 < d_1$. Assume inductively that $d_n < d_{n+1}$. Then

$$d_{n+2} = (b/V)(1/P(d_{n+1})) > (b/V)(1/P(d_n)) = d_{n+1}, \tag{17}$$

since $P(d)$ is a decreasing function. ■

Corollary 1. *The successive approximations of the market price of the risky discount bond debt F_n are bounded above by the par value b.*

Proof: From (15c) we have $F_n = b(P(d_n)/P(d_{n-1}))$. ■

Theorem 2. *The sequence of quasi debt-to-asset ratios $\{d_n\}$ is convergent.*

Proof: Let $H(x) = (b/V)(1/P(x))$. Then (15b) gives $d_n = H(d_{n-1})$. We next demonstrate that

$$\lim_{x\to\infty} H'(x) = b/V < 1. \tag{18}$$

Now $H'(x) = (b/V)(-P'(x)/P^2(x))$. Equation (11) gives $P'(x) = -N(h_1)/x^2$, so we have $H'(x) = (b/V)N(h_1)/(x^2P^2(x))$. But $xP(x) = P(1/x)$ by Lemma 1, so $H'(x) = (b/V)N(h_1)/P^2(1/x)$. Now $P(1/x) \to P(0) = 1$ and $N(h_1) \to N(\infty) = 1$ as $x \to \infty$, so (18) follows. Therefore, there is an $M > 0$ such that the increasing function H is a contraction map on $[M, \infty)$. Either M is an upper bound for the increasing sequence $\{d_n\}$ or all but finitely many d_n are contained in the interval $[M, \infty)$. In the latter case $\{d_n\}$ converges to the unique fixed point of H on that interval. ■

Corollary 1. *The sequence of successive market price iterations $\{F_n\}$ converges to the par value of the bond.*

Proof: Note that $F_n = b(P(d_n)/P(d_{n-1})) \to b$ as $n \to \infty$. ■

This completes the proof of our main objective, but there remains one further question: Is the sequence of market price iterations $\{F_n\}$ strictly increasing? This conjecture is financially plausible, but it is not a consequence of our development so far. From the fact that $\{d_n\}$ is increasing we can conclude that both $\{R_n\}$ and $\{B_n\}$ increase, but not that $\{F_n\}$ increases. To establish this fact we need to generalize Merton's formula (11) for the derivative of $P(d)$.

Lemma 2. *Define $w(x) = (b/V)(1/P(x))$. The function $G(x) = bP(w(x))/P(x)$ is increasing on $[0, \infty)$ with range $[bP(b/V), V)$.*

Proof: We have $G'(x)/b = \{P(x)P'(w)w'(x) - P(w)P'(x)\}/P^2(x)$, where $w'(x) = -(b/V)P'(x)/P^2(x) = -wP'(x)/P(x)$. Thus, $G'(x)/b = P'(x)\{-wP'(w) - P(w)\}/P^2(x)$. From (11), we obtain $P'(x) = -N(h_1(x))/x^2$ and $P'(w) = -N(h_1(w))/w^2$ or $-wP'(w) = N(h_1(w))/w = P(w) - N(h_2(w))$ from the definition of P, the price of a dollar of risky debt. This gives the desired generalization of equation (11)

$$G'(x) = bN(h_1(x))N(h_2(w))/(x^2P^2(x)). \tag{19}$$

Thus G is increasing (this is the only part of the lemma that we will use). Also, $G(0) = bP(w(0))/P(0) = bP(b/V)$. L'Hospital's rule can be used to show that $\lim_{x\to\infty} G(x) = V$. ■

Theorem 3. *The sequence of market price iterations $\{F_n\}$ is increasing.*

Proof: We have $F_n = bP(d_n)/P(d_{n-1})$, since $d_n = (b/V)(1/P(d_{n-1}))$. Using the notation of Lemma 2, $d_n = w(d_{n-1})$ so $F_n = bP(w(d_{n-1}))/P(d_{n-1})$, i.e., $F_n = G(d_{n-1})$. Since G is increasing, the theorem is proved.

4. APPLICATION. Table 2 expresses the *generalized risk premium*, i.e., the amount by which the limit of the sequence $\{R_n\}$ exceeds $R_1 = \Omega$, the riskless rate of return, as a function of σ (row) and the ratio b/V (column) for a time of five years to maturity. It was computed by a simple BASIC program with a fixed number of iterations. We use it in the following two applications:

TABLE 2 RISK PREMIUM IN % TO SELL BOND AT PAR
5 YEAR MATURITY

	Bond at Par as % of Assets								
Sigma	10%	20%	30%	40%	50%	60%	70%	80%	90%
0.1	0.000	0.000	0.000	0.000	0.002	0.022	0.134	0.517	0.641
0.2	0.000	0.001	0.018	0.107	0.356	0.868	1.784	3.374	6.422
0.3	0.003	0.080	0.363	0.937	1.858	3.211	5.156	8.056	13.024
0.4	0.084	0.575	1.505	2.846	4.623	6.924	9.954	14.186	21.049
0.5	0.479	1.797	3.620	5.870	8.582	11.878	16.018	21.591	30.300
0.6	1.437	3.890	6.750	9.985	13.677	17.994	23.261	30.179	40.648
0.7	3.118	6.913	10.899	15.168	19.868	25.225	31.629	39.887	52.005
0.8	5.615	10.892	16.063	21.398	27.128	33.538	41.086	50.67	64.305
0.9	8.982	15.839	22.238	28.664	35.439	42.912	51.607	62.489	77.504

A: There are times when it is useful to estimate the volatility of the assets of a firm whose securities are not publicly traded. Suppose a small firm has a bank loan that is 20% of its assets. The loan is for five years at 1.875% over prime, and prime is 2% over the stripped five-year Treasury zero coupon note, a 3.875% risk premium. From Table 2 we estimate σ to be .6.

B: Suppose a takeover is to be financed by a large subordinated bond (a *junk bond*). If the current debt at par is 40% of assets, an additional 20% debt at par is needed to fund the takeover, and the firm has a $\sigma = .3$ with all debt maturing in five years, then what is the generalized risk premium on the new debt? From Table 2 we see that the respective risk premiums with $b/V = 40\%$ is .00937; when $b/V = 60\%$ it is .03211. Then $60(\Omega + .03211) = 40(\Omega + .00937) + 20(\Omega + x)$ or $x = \{(60)(.03211) - (40)(.00937)\}/20$, which gives a risk premium of about 7.76%.

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FRANKLIN LOWENTHAL is Professor of Accounting and Computer Information Systems at California State University, Hayward. He received his Ph.D. in mathematics at Stanford University under the late Charles Loewner. His early research concerned the classical Lie groups; more recently, his interests have been in operations research and taxation. He is a Silver Life Master at duplicate bridge and has signed the summit registers on almost all Class 1 peaks in the Sierras.
School of Business and Economics, California State University, Hayward, Hayward, CA 94542
flowenth@csuhayward.edu

ARNOLD LANGSEN is Professor Emeritus in Financial Economics at California State University, Hayward. In 1985, the Trustees of the California State Universities named him Outstanding Professor of the Year. He earned his Ph.D. at Georgia State University in 1973. Following service in WWII, he taught mathematics at the Ohio State University while an undergraduate student himself.
California State University, Hayward, Hayward, CA 94542

CLARK T. BENSON is a Senior Level Expert in the Division of Mathematics Research at the National Security Agency. His past life included a long stint as Professor of Mathematics at the University of Arizona. He is interested in all aspects of communications theory, especially cryptology.
Division of Mathematics Research, National Security Agency, Ft. George G. Meade, MD 20755
ctbenso@afterlife.ncsc.mil